APPLICATION OF THE METHOD OF INTEGRAL DIFFUSION TO INVESTIGATION OF TURBULENT TRANSFER

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A number of authors have critically examined semiempirical mixing length theories [1]. A defect of these theories is connected with the fact that the magnitude of the mixing length, which is assumed to be small in constructing the theory, turns out in experiments to be comparable with the characteristic dimensions of the flow region. Thus, the concept of "volume convection" [2-4] or "integral diffusion" [5], which is understood to be a transfer mechanism in which the friction stress is not expressed in terms of the velocity gradient, is introduced along with the concept of "gradient diffusion." In addition, there are a number of experimental papers [6] in which it is shown that the turbulent friction stress cannot be equal to zero at the place in the flow where the derivative of the velocity is equal to zero. "Mixing length" theory does not describe this effect.

It is possible to generalize mixing length theory [7-9] in a way which eliminates these defects. Flow of an incompressible fluid is considered.

1. **Model of turbulent transfer.** The following flow scheme is adopted. As a result of instability, eddies are generated which have dimensions on the order of the characteristic dimension of the flow in this flow region. As the generated eddy, or "mole" is displaced in the flow, it is "stripped" by the environment, with the result that there is an exchange of momentum and heat. This picture of eddies (moles) appearing in layers with a large velocity gradient and penetrating the flow is widespread.

We shall assume that the generation of a mole is characterized by the local Reynolds number introduced by L. G. Loitsyanskii [10], which is made up of the characteristic rate of generation of pulsating motion $C^{*\prime}$, the characteristic dimension l', identified later with the "mixing length" or with the "free path" of the mole, and the viscosity of the fluid ν

$$R^+ = l'C_*' / \nu.$$

If this number is greater than some quantity R_0^+, then moles will be generated in the flow.

The reasoning in regard to the existence of a critical value R_0^+ in connection with the dimensions of the "laminar" sublayer is given in [10-12]. The layer of the flow in which "moles" are generated is characterized by large values of the velocity gradient. In this layer, the free path l' may be quite small; then such a layer can be called "equilibrium," and the Prandtl theory of mixing length will be valid. Moles will fly out of this layer so that their interaction with the medium creates transfer of momentum and heat. Their propagation is characterized by a certain probability of interaction between two points and the friction stress or the heat flux density is characterized in the general case by integrals over the entire volume from the sources of "moles," with the corresponding probabilities taken into consideration.

Let us consider a unit area about some point in the flow M_0 oriented perpendicular to the y-axis and moving at the speed of averaged motion at this point [7] (later we shall make use of the modified scheme of N. I. Buleev for constructing the turbulent stress tensor). The mole which has left the neighborhood of point M_0 creates the following pulsations in passing through this area:

$$u' \approx C_*' \cos(s, x) + u(M) - u(M_0),$$
$$v' \approx C_*' \cos(s, y) + v(M) - v(M_0). \tag{1.1}$$

It is assumed that the moles fly out evenly in all directions from the point M.

The passing mole contributes to the friction stress (per unit mass)

$$d(u'v') = [C_*'\cos(s,x) + u(M) - u(M_0)][C_*'\cos(s,y) + v(M) - v(M_0)] \exp\left(-\int_{M_0}^{M} \frac{ds}{l'}\right)\frac{ds}{l'}, \tag{1.2}$$

where the factor in the form of an exponent characterizes the probability that the mole will move from one point M to point M_0, or the value of the part of the mole which reaches the point, or the value of the interaction of the mole with the environment.

The total value of the friction force is obtained from summing the contributions made by all "moles"

$$\rho\langle u'v'\rangle = \frac{\rho}{4\pi}\int\int_{M_0}^{\infty} d(u'v')\,d\Omega, \tag{1.3}$$

the other components of the tensor are written in a like manner.

When $l' \to 0$, we can pass to the limit, expanding the quantities u(M) and v(M) in Taylor series. We obtain

$$\langle u'v'\rangle = -\frac{C_*'l'}{3}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\right),$$

$$\langle u'u'\rangle = \frac{C_*'^2}{3} - 2\frac{C_*'l'}{3}\frac{\partial u}{\partial x} \quad \text{etc.}$$

that is, we arrive at the usual form of the connection between the additional stresses and the strain rate tensor coinciding with the assumption of isotropicity of the quantity l' [13].

The model considered here (1.3) is also called integral diffusion. This model includes anisotropy of transfer in different directions, as the probability of interaction depends on the "optical thickness"

$$\int_0^s \frac{ds}{l'}$$

which is different along rays going in different directions.

The value of the friction stress is not directly connected with the velocity gradient. If the velocity profile is not symmetric, the frictional stress is not necessarily equal to zero at that point in the flow where the derivative of the velocity is zero.

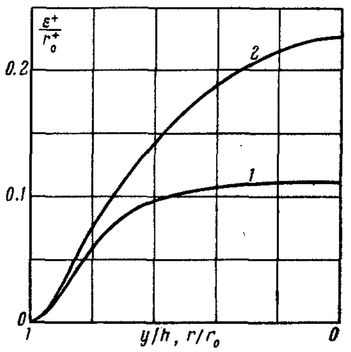

Fig. 1. Distribution of ε^+/r_0^+ across the channel.

In the plane case, the expressions for the friction stress and heat flux density are of the form

$$\tau_{xy} = \rho \int_{y_0}^{\infty} \frac{C_*[u(y)-u(y_0)]}{2} E_2\left(\int_{y_0}^{y} \frac{dy}{l'}\right) \frac{dy}{l'} + \rho \int_{-\infty}^{y_0} \ldots,$$

$$q_y = \rho \int_{y_0}^{\infty} \frac{C_* C_p [T(y)-T(y_0)]}{2P_1} E_2\left(\int_{y_0}^{y} \frac{dy}{l'}\right) \frac{dy}{l'} + \rho \int_{-\infty}^{y_0} \ldots \quad (1.4)$$

where the turbulent Prandtl number P_1 is introduced.

The expression (1.4) can be simplified if we average over the angle $\overline{\cos\theta} = 2/3$,

$$\tau_{xy} = \rho \int_{-\infty}^{y} [u(\xi) - u(y)] \frac{C_*}{4} \exp\left(-\int_{\xi}^{y} \frac{dz}{l}\right) \frac{d\xi}{l} - \rho \int_{y}^{\infty} [u(\xi) - u(y)] \frac{C_*}{4} \exp\left(-\int_{y}^{\xi} \frac{dz}{l}\right) \frac{d\xi}{l} \quad \text{etc.} \quad (1.5)$$

In many cases, we may not be interested in effects produced by flow asymmetry, and can simplify the obtained expressions, expanding the differences Δu and ΔT into series [7]

$$\tau_{xy} \approx -\rho\varepsilon \frac{du}{dy}, \qquad q_y \approx -\frac{\rho C_p \varepsilon}{P_1} \frac{dT}{dy}, \quad (1.6)$$

$$\varepsilon = \int_{-\infty}^{+\infty} \frac{C_*}{4} |\xi - y| \exp\left(-\left|\int_{\xi}^{y} \frac{dz}{l}\right|\right) \frac{d\xi}{l}. \quad (1.7)$$

2. Values of the "mixing length" and rate of generation. There are "moles" of various sizes at every point in the flow. In principle, the problem can be complicated by introducing distribution according to scales. However, we shall restrict ourselves to considering the most characteristic dimension as being somehow connected with the properties of the distribution of the velocity field or with the characteristic dimension of the "generating" layer in the free jet and with the distance to the wall bounding the flow, which regulates the size of the moles in the flow.

We shall take the relationship proposed by A. M. Obukhov [14] as the scale l. It is assumed that the scale distribution is determined only by the geometrical properties of the channel and that the hypothesis of local similarity is utilized. At this point, we introduce the interior flow geometry, an element of which $d\sigma = l^{-1} ds$ coincides with the quantity $l^{-1} ds$ characterizing the probability of interaction introduced in the model under consideration. The author also pointed out the flow anisotropy appearing in this connection.

The scale distribution is determined with accuracy to a constant in the following cases:

$$l = ky \quad \text{for the half-plane,}$$

$$\frac{l}{h} = \frac{2k}{\pi} \cos\left(\frac{\pi z}{2h}\right) \quad \text{for a plane slit with the width 2h,}$$

$$\frac{l}{r_0} = \frac{k}{2}\left[1 - \left(\frac{r}{r_0}\right)^2\right] \quad \text{for a circular pipe of radius } r_0. \quad (2.1)$$

Here r and z are the distances from the axis and the middle plane. These distributions agree quite well with the generally used expressions for the "mixing length."

On comparing the expression for τ_{xy} in the generating layer when $l \to 0$ with the expression in accordance with Prandtl's mixing length theory, we find that the quantity C_* is proportional to the friction velocity V_*. We shall accept as a hypothesis that the following relationship is valid for C_*:

$$C_* = C_0 \sqrt{|\tau_T|/\rho} \qquad (C_0 = \text{const}).$$

Here τ_T is the Reynolds frictional stress. This expression can be extended to the three-dimensional case.

Bearing in mind that "mixing length" theory describes turbulent transfer in the generating layer sufficiently well, one may expect this assumption to yield satisfactory results in the integral diffusion model also.

The hypotheses enumerated here on the relationship for l, on the relationship for C_*, on the existence of the critical local Reynolds number R_0^+, and the use of the turbulent Prandtl number P_1 are sufficient to close the system of equations.

The constants k, C_0, R_0^+, and P_1 introduced here should be determined from experimental data.

3. Boundary layer on a flat plate. Close to the wall, where the mixing length is short, the value of the frictional stress is approximately constant.

Then

$$\frac{\tau_{xy}}{\rho} = -V_*^2 (1 + \varepsilon^+) \frac{du^+}{dy^+} = -V_*^2,$$

$$y^+ = \frac{yV_*}{\nu}, \qquad u^+ = \frac{u}{V_*}.$$

Here y is the distance from the wall (l = ky), and the rate of generation is constant when $y^+ > y_0^+$ and equal to zero when $y^+ < y_0^+$.

In accordance with this, we obtain the following expressions for ε^+:

$$\frac{\varepsilon^+ 4}{c_0} = ky^+\left[\left(1 - \left(\frac{y_0^+}{y^+}\right)^{1/k}\right)\frac{1}{1+k} + \frac{1}{1-k}\right] \text{ when } y^+ > y_0^+ \quad (3.1)$$

$$\varepsilon^+ \to \frac{2kC_0}{(1-k^2)4}\, y^+ \quad \text{when} \quad \frac{y^+}{y_0^+} \gg 1 \tag{3.2}$$

$$\frac{\varepsilon^+ 4}{c_0} = \frac{y_0^+}{1-k}\left[\left(\frac{y^+}{y_0^+}\right)^{1/k} - (1-k)\left(\frac{y^+}{y_0^+}\right)^{1/k+1}\right] \text{when } y^+ < y_0^+ \tag{3.3}$$

$$\varepsilon^+ \to \frac{y_0^+}{1-k}\,\frac{C_0}{4}\left(\frac{y^+}{y_0^+}\right)^{1/k} \quad \text{when} \quad \frac{y^+}{y_0^+} \ll 1. \tag{3.4}$$

There are data (refer, for example to [15, 16]) on the behavior of the turbulent heat conductivity:

$$\frac{\varepsilon^+}{P_1} = a^4 (y^+)^4 \quad \text{when } y^+ \to 0.$$

Thus, $a = 0.124$ according to Deissler, $a = 0.09$ according to Spalding; it is accepted below that $a = 0.106$.

Comparing formula (3.4) for ε^+ with this relation and formula (3.2) for ε^+ with the relation $\varepsilon^+ = 0.40\, y^+$, we obtain

$$k = 0.25, \quad C_0 = 3, \quad y_0^+ = 20.$$

The value of P_1 was taken to be equal to 0.71.

The dependence of ε^+ on y^+, the velocity profile in the boundary layer, and the data on heat transfer at large values of the Prandtl number agree satisfactorily with the data presented in [15]. The constants determined by these data will be used in solving other problems.

4. Flow in a flat channel and Couette flow. The distribution of the mixing length is given by formula (2.1). For flow in a flat channel

$$\frac{\tau}{\tau_w} = \frac{y}{h}, \qquad \frac{\tau_T}{\tau_w} = \frac{\varepsilon^+}{1+\varepsilon^+}\,\frac{y}{h}.$$

Making use of this expression and formulas (3.1) and (3.3) for ε^+, we find the distribution of ε^+ across the channel (Fig. 1, curve 1 ($h^+ = 250$)) and the velocity distribution which agrees well with the distribution $u^+ = 5.1 + 2.5 \ln y^+$.

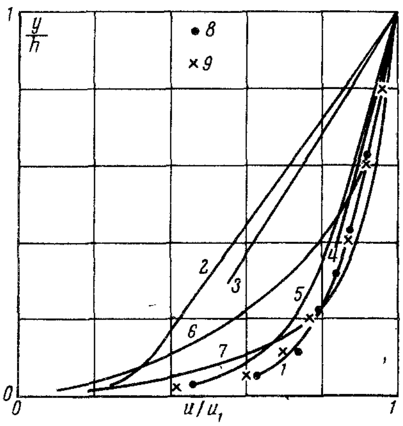

Fig. 2. Velocity distribution: 1) Karman (1937)—with accuracy to the constant $Cf = 0.0042$; 2) Pai (1952); 3) Munk (1955); 4) Reichardt (1955); 5) Robertson (1957); 6) Wu (1958); 7) Wu-Robertson (1958); 8) experimental data of [17]; 9) curve from calculations by the present method.

The problems considered up to this time are also solved by making use of the usual mixing length theory with corresponding choice of the constants and the given relationship $\varepsilon^+(Y^+)$. Thus, in these cases, we merely verified the fact that the integral diffusion model produces a satisfactory solution.

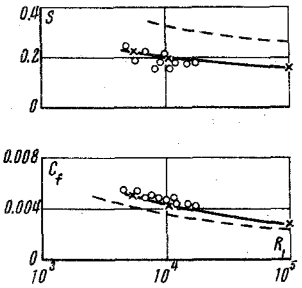

Fig. 3. The derivatives of the velocity profile and of the resistance coefficient (solid curves—calculations by the present method, dashed curves—in accordance with the mixing length theory; 0 are experimental points, data from [17]).

Below we shall consider turbulent Couette flow which the mixing length theory with constants chosen for the problem of the boundary layer does not solve satisfactorily—the constant characterizing the value of turbulent viscosity must be almost doubled [17].

This problem was solved as an approximation to integral diffusion. The turbulent friction stress close to the wall is determined by the formula

$$\frac{\tau_T}{\tau_w} = \frac{\varepsilon^+}{1+\varepsilon^+},$$

the value of ε^+ is again determined by formulas (3.1) and (3.3); it is given in Fig. 1 (curve 2, $h^+ = 250$). In the core of the flow, the value of ε^+ for Couette flow is almost double the value of ε^+ for flow in a flat channel. The difference appears in these cases in the different distribution of the generating rate C^* in the transverse section.

Having the distribution ε^+, we can obtain an expression for the slope of the velocity at the center

$$S = \frac{h^+}{u_1^+}\left(\frac{du^+}{dy^+}\right)_1$$

and the resistance coefficient

$$C_f = \frac{\tau_w}{{}^1\!/_2\, \rho U_1^2} = \frac{2}{(u_1^+)^2}$$

from $R_1 = u_1^+ h^+$, where U is the value of the flow velocity on the axis.

Figure 2 shows the velocity distribution when $R_1 = 10^4$ and presents a comparison of the obtained dependence with experimental data and different solutions [17]. Figure 3 presents a comparison of the relationships S and Cf.

It can be seen from a comparison of the solution obtained here with experimental data that the agreement of the velocity profile and the value of S is sufficiently good, and the value of Cf obtained in the calculations was 10% less than that from the experimental relationship. If we recall that the values of the constants did not change, this agreement can be considered satisfactory.

REFERENCES

1. G. K. Batchelor, "Note on free turbulent flows with special reference to the two-dimensional wake," J. Aeronaut. Sci., vol. 17, no. 3, pp. 441–445, 1950.

2. A. A. Townsend, Structure of Turbulent Shear Flow [Russian translation], Izd. inostr. lit, 1959.

3. G. B. Schubauer and K. M. Cheng, "Turbulent flow," in: Turbulent Flows and Heat Transfer [Russian translation], edited by C. C. Lin, chapter II, Izd. inostr. lit, pp. 83–205, 1963.

4. J. O. Hinze, Turbulence [Russian translation], Fizmatgiz, 1963.

5. I. C. Schönfeld, "Integral diffusivity," J. Geophys. Res., vol. 67, no. 8, pp. 3187–3199, 1962.

6. W. H. Schwarz and W. P. Cosart, "The two-dimensional turbulent wall-jet," J. Fluid Mech., vol. 10, part 4, pp. 481–495, 1961.

7. N. I. Buleev, "A theoretical model of the mechanism of turbulent exchange in fluid flows," in: Heat Transfer [in Russian], Izd-vo AN SSSR, pp. 64–98, 1962.

8. A. T. Onufriev, "A model of nonequilibrium processes in some problems of continuum mechanics," PMTF, no. 1, pp. 47–56, 1963.

9. E. A. Spiegel, "A generalization of the mixing-length theory of turbulent convection," Astrophys. J., vol. 138, no. 1, pp. 216–225, 1963.

10. L. G. Loitsyanskii, "Turbulent motion of a fluid and the internal problem," Izvestiya NII gidroteckhniki, vol. 9, pp. 1–30, 1933.

11. M. A. Velikanov, Dynamics of Channel Flows [in Russian], part I, Gostekhizdat, 1954.

12. F. Klauser, "The turbulent boundary layer," in: Problems of Mechanics [Russian translation], Izd. inostr. lit., no. 2, pp. 297–340, 1959.

13. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], part II, Izd-vo "Nauka," 1965.

14. A. M. Obukhov, "The distribution of the scale of turbulence in flows of arbitrary section," PMM, vol. 6, no. 2–3, pp. 209–220, 1942.

15. D. B. Spalding, "Heat transfer to an isothermal flat plate in a turbulent flow," Inzh.-fiz. zh., no. 3, 1963.

16. D. B. Spalding, "Investigation of the heat transfer of an isothermal flat plate with turbulent fluid flow," Inzh.-fiz. zh., no. 3, 1963.

17. I. M. Robertson, On Turbulent Plane-Couette Flow, Proc. of the Sixth mid-western Conference of fluid mechanics, The University of Texas, USA, pp. 169–182, 1959.

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